

Remarks on modules over deformation quantization algebras

Ryszard Nest and Boris Tsygan

To Boris Feigin on his fiftieth birthday

1. INTRODUCTION

The aim of this paper is to provide a link between deformation quantization theory of [BFFLS] and [Fe] and Lagrangian analysis. By the latter we mean Maslov's theory of canonical operator [M], [MSS], [NSS], [L] and Hörmander's theory of distributions given by oscillatory integrals [GS], [H]. Though it was always clear that such links exist (cf., for example, [Kar]), and though the creators of deformation quantization were probably partially motivated by Lagrangian analysis, we are not aware of any works that relate the two subjects explicitly.

Here are three reasons why, in our view, such a link may be desired. First of all, there is a pedagogical reason: it is natural to look for a more unified approach to the two important subjects that are clearly related. Secondly, there is a motivation from index theory. Namely, one can try to extend the Atiyah-Singer index theorem from pseudo-differential operators to a more general class of so called Guillemin-Sternberg operators which are Fourier integral operators of special kind [GS1], [GU]. (This is the authors' joint project with A. Gorokhovsky). To prove such a theorem, one would try to reduce it to a general index theorem for deformation quantization like in [BNT], [NT2]. For that, one needs to answer questions which are studied in the present paper: how to relate Lagrangian analysis to deformation quantization and, more precisely, how to pinpoint the resulting algebra in terms of the general classification of deformation quantizations of a symplectic manifold given by the construction of Fedosov.

Finally, our third motivation is related to mirror symmetry. It is a general feeling among the experts that the Fukaya category of a symplectic manifold is somehow, in a very nontrivial way, related or analogous to the derived category of modules over a deformation quantization of this manifold (cf., for example, [BS] or [KS]; the idea that deformation quantization should be related to Lagrangian intersection theory was communicated to the second author by Boris Feigin in the mid 80s). We think that our constructions may suggest new structures on modules over deformed algebras, which would lead to modified versions of the derived category of modules which might be related to the Fukaya category somewhat more closely.

More precisely, when our symplectic manifold has the first Chern class equal to zero, then its deformation quantization leads to an additional structure, namely to a groupoid \tilde{G} which, roughly speaking, consists of expressions $\exp(\frac{i}{\hbar}H)$ where H is a function (cf. 4.6). Our feeling is that the kinds of modules which appear in deformation theory from Lagrangian analysis are something like objects of a new derived category of complexes of locally free modules; in that new category, localizing with respect to quasi-isomorphisms is modified so that elements of \tilde{G} are included into the set of quasi-isomorphisms by which we allow to localize. Such a construction could be a somewhat better approximation to the Fukaya category because it is not local, i.e. not purely sheaf-theoretical. Also, those modules would be in a closer relation to the Maslov phenomena which are central to Lagrangian intersections [Se] but cannot be seen by the ordinary homological algebra of modules over deformation quantizations.

Let us describe the contents of the paper in more detail. After some preliminaries on Lagrangian subspaces, Lagrangian submanifolds, the Maslov index, and an algebraic version of the Weil representation, we review the Fedosov construction and classification of deformation quantizations. Then we remind how to construct a deformation quantization of T^*X starting from differential operators on X . We then compute this deformation quantization in terms of Fedosov's classification [Fe]. Our version of this construction essentially follows [BNT], but we design a modified Fedosov construction which streamlines the exposition. More precisely, the Fedosov construction provides a deformation quantization starting from a multiplication preserving connection on the Weyl bundle of a symplectic manifold. We show that differential operators on half-densities lead to a product which is defined directly on the bundle of jets. This product is preserved by the canonical connection on the jet bundle. This is the canonical bundle of algebras \mathcal{W} which is isomorphic to the Weyl bundle W of Fedosov. This isomorphism is canonical up to a canonical connection-preserving inner automorphism. (This means, here and below, that the isomorphism canonically depends on a choice of an auxiliary datum; isomorphisms corresponding on two different choices differ by a conjugation by an element which is canonically constructed from the pair of data. Next we review Lagrangian analysis, in particular Hörmander's construction of distributions whose wave front is a given Lagrangian submanifold L . We show that, after an extension of the ring of scalars, the asymptotics

of this construction leads to a module over the deformation quantization of T^*X discussed above. Our exposition here is close to Maslov's method of canonical operator, cf. [NSS].

We would like to express this module in more familiar deformation-theoretic terms. There are two equivalent ways of doing that. First, one can express it in terms of the Fedosov construction of deformations of symplectic manifolds. Second, we can apply Darboux-Weinstein theorem and identify an open neighborhood of L in T^*X with an open neighborhood of L in T^*L . By the classification theorem for deformations of a symplectic manifold, the deformed algebra on T^*X , restricted to the neighborhood, becomes isomorphic to the standard deformation on T^*L . Moreover, as we show in 8.1.1, this isomorphism is canonical up to a canonical inner automorphism. We prove that, after identifying the two deformed algebras using this isomorphism, the Lagrangian module corresponding to L in T^*X becomes isomorphic to the similar module corresponding to L in T^*L , tensored by the flat bundle given by a certain Čech one-cocycle (Theorem 8.1.2). This cocycle involves the Maslov class of L and the cohomology class of $\alpha|_L$ where α is the standard one-form on T^*X such that $d\alpha = \omega$.

Our key observation is that modules of the type we consider are still perfectly well described by Gelfand's formal differential geometry. Namely, one can construct the bundle of jets of sections of such a module, which is a bundle of modules over the algebra of jets of functions. This is, in a sense, a second microlocalization: after having localized the distributions to a Lagrangian submanifold, we now further localize them to any point of this submanifold.

Once the jet formalism for our modules is established, one can compare them to each other. More precisely, using local phase functions of L , we see that the Lagrangian jet bundle is isomorphic to the vector bundle induced by the algebraic Weil representation of the universal cover of the symplectic group. The isomorphism is given, essentially, by the Maslov canonical operator at the jet level. This bundle, in turn, is easy to compare to the Lagrangian jet bundle of L in T^*L .

We believe that most of the contents of the paper are well known to experts in some form. The second author is greatly indebted to Boris Feigin for introducing him to the topic (and to deformation quantization in general, as well as to many other things). He is also grateful to Alexander Karabegov for sharing a key idea how to establish a direct connection between the stationary phase method and deformation quantization. We are grateful to B. Sternin for a masterful exposition of the theory of canonical operator. We are grateful to D. Arinkin, A. Beilinson, R. Bezrukavnikov, P. Bressler, V. Drinfeld, K. Fukaya, D.

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2. PRELIMINARIES FROM SYMPLECTIC LINEAR ALGEBRA

2.1. Let \mathbb{R}^{2n} be the standard symplectic vector space with the symplectic form $d\xi \wedge dx = \sum d\xi^k \wedge dx^k$. Let Λ be the set of Lagrangian subspaces of \mathbb{R}^{2n} . One has $U(n)/O(n) \xrightarrow{\sim} \Lambda$. It is well known that the map

$$u \in U(n) \mapsto \det(u)^2 \in S^1$$

induces an isomorphism $\pi_1(\Lambda) \xrightarrow{\sim} \pi_1(S^1) = \mathbb{Z}$.

For $N \in \mathbb{Z}$, $N > 0$, put

$$\tilde{U}^N(n) = \{(u, \zeta) | u \in U(n), \zeta \in \mathbb{C}, \det(u)^2 = \zeta^N\}$$

This is a central extension of U by \mathbb{Z}/N . The space $\tilde{\Lambda}^N = \tilde{U}^N/O$ is a cover of Λ with the deck transformation group \mathbb{Z}/N . Put also

$$\tilde{U}(n) = \{(u, x) | u \in U(n), x \in \mathbb{R}, \det(u)^2 = \exp(2\pi i x)\}$$

The space $\tilde{\Lambda} = \tilde{U}/O$ is the universal cover of Λ .

Let us describe a Čech one-cocycle determining the covering $\tilde{\Lambda} \rightarrow \Lambda$. For $I \subset \{1, \dots, n\}$, let $x_1 = (x^k | k \in I)$, $x_2 = (x^k | k \in \bar{I})$, $\xi_1 = (\xi^k | k \in I)$, $\xi_2 = (\xi^k | k \in \bar{I})$ where \bar{I} is the complement of I . Let L_I be the Lagrangian subspace $\{x_2 = 0, \xi_1 = 0\}$. Let U_I be the open subset of those L for which the projection onto L_I along $L_{\bar{I}}$ is an isomorphism. In other words, U_I consists of those L which are defined by equations

$$\xi_1 = Ax_1 + B\xi_2 \tag{2.1}$$

$$x_2 = -B^t x_1 - C\xi_2 \tag{2.2}$$

where A, C are self-adjoint matrices and t means transposition.

Now let us describe intersections $U_I \cap U_J$. Let $I = I_3 \cup I_4$, $J = I_2 \cup I_4$, where I_p , $p = 1, \dots, 4$, are disjoint and cover $\{1, \dots, n\}$. Put $x_p = (x^k | k \in I_p)$ and $\xi_p = (\xi^k | k \in I_p)$. A Lagrangian subspace L is in $U_I \cap U_J$ if and only if it can be described by equations

$$\begin{aligned} \xi_1 &= \dots \\ \xi_2 &= \dots + Ax_2 + B\xi_3 + \dots \\ x_3 &= \dots - B^t x_2 - C\xi_3 + \dots \\ x_4 &= \dots \end{aligned}$$

where the matrix $\begin{bmatrix} A & B \\ {}^t B & C \end{bmatrix}$ is nondegenerate.

Theorem 2.1.1. ([H], [GS]). *The formula*

$$c_{IJ} = \frac{1}{2} \text{signature} \begin{bmatrix} A & B \\ B^t & C \end{bmatrix}$$

defines a $\frac{1}{2}\mathbb{Z}$ -valued Čech 1-cocycle for the cover $\{U_I\}$. This cocycle is cohomologous to a \mathbb{Z} -valued 1-cocycle whose cohomology class is the class of the universal cover of Λ .

Here by the signature of a self-adjoint nondegenerate matrix we mean the signature of the corresponding quadratic form.

There is a central extension $\widetilde{\text{Sp}}^N$ of Sp by $\mathbb{Z}/N\mathbb{Z}$ of which \tilde{U}^N is a subgroup. It is defined by

$$\widetilde{\text{Sp}}^N = \{(g, \gamma)\}$$

where $g \in \text{Sp}$ and γ is a homotopy class of a path in Λ connecting L_0 and $g(L_0)$. Similarly, one constructs a central extension $\widetilde{\text{Sp}}$ of Sp by \mathbb{Z} .

2.2. The Weil representation. Here we recall in a slightly more algebraic form the standard construction of the Weil representation (cf., for example, [L] and [GS]). Let V^{Weil} be the vector space

$$V^{\text{Weil}} = \bigoplus_T \exp\left(\frac{iT\hat{x}^2}{2\hbar}\right) \mathbb{C}[[\hat{x}^1, \dots, \hat{x}^n, \hbar]],$$

i.e. a free $\mathbb{C}[[\hat{x}^1, \dots, \hat{x}^n, \hbar]]$ -module with the basis indexed by all complex symmetric $n \times n$ matrices T such that $\text{Im } T$ is positive definite. The group $\widetilde{\text{Sp}}^4$ acts on V^{Weil} as follows. Let

$$F(\hat{x}) = \exp\left(\frac{iT\hat{x}^2}{2\hbar}\right) f(\hat{x}^1, \dots, \hat{x}^n, \hbar)$$

where f is a formal power series. Then

$$\begin{bmatrix} 1 & A \\ 0 & 1 \end{bmatrix} : F(\hat{x}) \mapsto \exp\left(\frac{iA\hat{x}^2}{2\hbar}\right) F(\hat{x})$$

for a real symmetric $n \times n$ matrix A ;

$$\begin{bmatrix} B & 0 \\ 0 & B^{-1t} \end{bmatrix} : F(\hat{x}) \mapsto |\det B|^{-\frac{1}{2}} F(B^{-1}\hat{x})$$

for $B \in \text{GL}(n, \mathbb{R})$;

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} : F(\hat{x}) \mapsto \text{Fourier} F(\hat{x});$$

the generator of the center of $\widetilde{\text{Sp}}^4$ acts by multiplication by the imaginary unit i . Here Fourier stands for the Fourier transform at the level

of power series as explained in [K]; cf. [L], [GS] for a related definition of the Fourier transform of asymptotics. Namely,

$$\text{Fourier exp}\left(\frac{iT\hat{x}^2}{2\hbar}\right)f(\hat{x}, \hbar) = f(\hat{\xi}, \hbar)\text{Fourier exp}\left(\frac{iT\hat{x}^2}{2\hbar}\right)$$

where $\hat{\xi}_k = i\hbar \frac{\partial}{\partial \hat{x}^k}$ and

$$\text{Fourier exp}\left(\frac{iT\hat{x}^2}{2\hbar}\right) = (\det(-iT^{-1}))^{-\frac{1}{2}} \exp\left(-\frac{iT^{-1}\hat{x}^2}{2\hbar}\right)$$

Since the imaginary part of T is positive definite, the square root is well defined.

There is also a degenerate version of the Weil representation:

$$V_0^{\text{Weil}} = \bigoplus_T \exp\left(\frac{iT\hat{x}^2}{2\hbar}\right) \mathbb{C}[[\hat{x}^1, \dots, \hat{x}^n, \hbar]],$$

where the sum is now taken over all real symmetric $n \times n$ matrices T . On this space, the representation is only partially defined. Namely, on the subspace $V_{T=0}$ one can define operators corresponding to elements of the open dense subset of $\widetilde{\text{Sp}}^4$ whose projection to Sp consists of matrices

$$\begin{bmatrix} 1 & A \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & B \\ B^{-1t} & 0 \end{bmatrix} \begin{bmatrix} 1 & C \\ 0 & 1 \end{bmatrix}$$

where A is nondegenerate. On the other hand, the operators corresponding to elements whose projection is equal to

$$\begin{bmatrix} B & 0 \\ 0 & B^{-1t} \end{bmatrix}$$

are defined everywhere. Those elements whose action is defined on the subspace corresponding to an arbitrary T belong to the above set conjugated by $\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$. Now one has to be more careful about the square root. We define for a real symmetric nondegenerate matrix T

$$(\det(iT))^{\frac{1}{2}} = \prod_{k=1}^n \sqrt{i\lambda_k}$$

where λ_k are the eigenvalues and we take the branch of the square root

$$\sqrt{re^{i\varphi}} = \sqrt{r}e^{i\varphi/2}$$

for $-\pi < \varphi < \pi$.

If the action of elements g , h , and gh are defined on a vector, then the latter is equal to the composition of the former two. This can be seen by passing to the limit $T \rightarrow 0$ from V^{Weil} to V_0^{Weil} .

3. PRELIMINARIES FROM SYMPLECTIC GEOMETRY

3.1. Let (M, ω) be a symplectic manifold. An $\widetilde{\mathrm{Sp}}^N$ -structure on M is by definition a reduction of the structure group of the tangent bundle T_M to $\widetilde{\mathrm{Sp}}^N$. The group $H^1(M, \mathbb{Z}/N)$ acts transitively and freely on the set of isomorphism classes of such reductions. An $\widetilde{\mathrm{Sp}}^N$ -structure on M exists if and only if the image of $2c_1(T_M)$ in $H^2(M, \mathbb{Z}/N)$ is equal to zero. Here $c_1(T_M)$ is the first Chern class of the tangent bundle viewed as a complex vector bundle (after choosing an almost complex structure compatible with the symplectic form).

An equivalent definition of a $\widetilde{\mathrm{Sp}}^N$ -structure is as follows. Let Λ_M be the bundle whose fiber over a point x is the Grassmannian of Lagrangian subspaces of $T_x M$. To give an $\widetilde{\mathrm{Sp}}^N$ -structure on M is the same as to give a bundle $\tilde{\Lambda}_M^N$ with fiber $\tilde{\Lambda}$, together with a morphism of bundles $\tilde{\Lambda}_M^N \rightarrow \Lambda_M$ which is, at the level of the fibers, the morphism $\tilde{\Lambda}^N \rightarrow \Lambda$ (cf. [Se]).

If a distribution of Lagrangian subspaces is given on M , then one can define an associated $\widetilde{\mathrm{Sp}}^N$ -structure as follows: the distribution provides a base point in every Λ_x , and one uses this base point to define $\tilde{\Lambda}_x^N$. In the language of transition functions, observe that a distribution by Lagrangian subsets defines a reduction of the structure group from Sp to the subgroup stabilizing a fixed Lagrangian submanifold \mathbb{R}^n of \mathbb{R}^{2n} . But this subgroup admits a canonical lifting to a subgroup of $\widetilde{\mathrm{Sp}}$, $g \mapsto (g, \gamma)$ where γ is the constant path.

Given a Lagrangian submanifold L of M , and given an $\widetilde{\mathrm{Sp}}^N$ -structure on M , one defines *the Maslov class* of L as follows. On a tubular neighborhood of L , there are two $\widetilde{\mathrm{Sp}}^N$ -structures. One is the restriction of the structure on M , the other comes from a Lagrangian distribution which is transverse to L (its isomorphism class does not depend on a choice of such distribution). The two differ by an element of $H^1(L, \mathbb{Z}/N)$ which we call the Maslov class. Similarly, for an $\widetilde{\mathrm{Sp}}$ -structure on M and for a Lagrangian submanifold L , we define the Maslov class in $H^1(L, \mathbb{Z})$.

Let us now describe the Maslov class of a Lagrangian submanifold of T^*X (where the $\widetilde{\mathrm{Sp}}$ -structure comes from the distribution consisting of tangent spaces to fibers of the projection $T^*X \rightarrow X$) by a Čech one-cocycle of L . Let $\pi : T^*X \rightarrow X$ be the projection. Consider an open cover $X = \cup U_\alpha^0$ and a refinement $T^*X = \cup U_\beta$ of the cover by $\pi^{-1}(U_\alpha^0)$. Every U_β is contained in some $\pi^{-1}(U_{\alpha(\beta)}^0)$. Using notation of

section 2, subdivide the coordinates on each $U_{\alpha(\beta)}^0$ into two groups,

$$x = (x_1, x_2) \quad (3.1)$$

in such a way that $L \cap U_\beta$ is given by equations

$$\xi_1 = F_{x_1}(x_1, \xi_2) \quad (3.2)$$

$$x_2 = -F_{\xi_2}(x_1, \xi_2) \quad (3.3)$$

For an intersection $U_\beta \cap U_\gamma$, our data can differ in two ways.

- 1) They may differ by a choice of coordinates on X .
- 2) They may differ by a choice of subdivision 3.1 for the same coordinate system.

Now let us start to define a one-cocycle representing the Maslov class. In case 1), put

$$c_{\beta\gamma} = 0 \quad (3.4)$$

In case 2), using notation of section 2, let $I = I_3 \cup I_4$ for U_β and $I = I_2 \cup I_4$ for U_γ . On the intersection, L can be described by equations

$$\xi_1 = F_{x_1}(x_1, x_2, \xi_3, \xi_4) \quad (3.5)$$

$$\xi_2 = F_{x_2}(x_1, x_2, \xi_3, \xi_4) \quad (3.6)$$

$$x_3 = -F_{\xi_3}(x_1, x_2, \xi_3, \xi_4) \quad (3.7)$$

$$x_4 = -F_{\xi_4}(x_1, x_2, \xi_3, \xi_4) \quad (3.8)$$

where the Hessian matrix $\text{Hess}_{\xi_2, x_3}(F)$ is nondegenerate. Put

$$c_{\beta\gamma} = \frac{1}{2} \text{signature } \text{Hess}_{\xi_2, x_3}(F) \quad (3.9)$$

Theorem 3.1.1. ([H], [GS]) *The cochain c is a $\frac{1}{2}\mathbb{Z}$ -valued Čech one-cocycle of L which is cohomologous to a \mathbb{Z} -valued cocycle representing the Maslov class.*

The proof will be contained in the proof of Theorem 8.1.2.

4. PRELIMINARIES FROM DEFORMATION QUANTIZATION

4.1. Let (M, ω) be a symplectic manifold. A *deformation quantization* of M (cf. [BFFLS]) is a formal power series

$$f * g = fg + \sum_{k=1}^{\infty} (i\hbar)^k D_k(f, g) \quad (4.1)$$

where $D_k : C^\infty(M) \otimes C^\infty(M) \rightarrow C^\infty(M)$ are bilinear bidifferential expressions, $*$ is associative, $f * 1 = 1 * f = f$, and

$$\{f, g\} = D_1(f, g) - D_1(g, f) \quad (4.2)$$

is the Poisson bracket defined by the symplectic structure.

Definition 4.1.1. An isomorphism between $*$ and $*'$ is a formal series

$$T(f) = f + \sum_{k=1}^{\infty} (i\hbar)^k T_k(f) \quad (4.3)$$

where $T(f * g) = T(f) *' T(g)$ and T_k are differential operators on $C^\infty(M)$.

Example 4.1.2. For $M = \mathbb{R}^{2n}$ and $\omega = d\xi \wedge dx$, put

$$f * g = \exp\left(\frac{i\hbar}{2}(\partial_\xi \partial_y - \partial_\eta \partial_x)\right) f(x, \xi) g(y, \eta)|_{x=y, \xi=\eta} \quad (4.4)$$

(the Moyal product). This is a deformation quantization of \mathbb{R}^{2n} .

We will denote by W (the Weyl algebra) the algebra $\mathbb{C}[[\hat{x}^1, \dots, \hat{x}^n, \hat{\xi}^1, \dots, \hat{\xi}^n, \hbar]]$ with the Moyal product (4.4) (we always denote formal variables by $\hat{x}, \hat{\xi}$). One can identify W with the ring of operators of the form $\sum A_{\alpha\beta} \hat{x}^\alpha (i\hbar \frac{\partial}{\partial \hat{x}})^\beta$ on $\mathbb{C}[[\hat{x}, \hbar]]$. This identification takes $\hat{x}^\alpha \hat{\xi}^\beta$ to the symmetrized product $\hat{x}^\alpha (i\hbar \frac{\partial}{\partial \hat{x}})^\beta$ (the Weyl identification). Note that, if one puts

$$|\hat{x}^k| = |\hat{\xi}^k| = 1, |\hbar| = 2, \quad (4.5)$$

then W becomes a direct product of its graded components

$$W = \prod_{k \geq 0} W_k$$

Put

$$\mathfrak{g} = \frac{1}{i\hbar} W / \frac{1}{i\hbar} \mathbb{C}[[\hbar]] \quad (4.6)$$

with the bracket $a * b - b * a$. This Lie algebra is isomorphic to the algebra of continuous derivations of W via $a \mapsto \text{ad}(a)$. The Lie algebra \mathfrak{g} splits into the product of its graded components

$$\mathfrak{g} = \prod_{k \geq -1} \mathfrak{g}_k$$

Note that $\mathfrak{g}_{-1} = \frac{1}{i\hbar} \mathbb{C}^{2n}$ and $\mathfrak{g}_0 = \mathfrak{sp}(2n, \mathbb{C})$. Also, the group $\text{Sp}(2n, \mathbb{C})$ acts on W by linear changes of coordinates. This action preserves the product. Its infinitesimal action coincides with the adjoint action of \mathfrak{g}_0 .

Put

$$\tilde{\mathfrak{g}} = \frac{1}{i\hbar} W \quad (4.7)$$

with the bracket $a * b - b * a$. One has

$$\tilde{\mathfrak{g}} = \prod_{k \geq -2} \tilde{\mathfrak{g}}_k$$

where $\tilde{\mathfrak{g}}_{-2} = \frac{1}{i\hbar}\mathbb{C}$, $\tilde{\mathfrak{g}}_{-1} = \frac{1}{i\hbar}\mathbb{C}^{2n}$ and

$$\tilde{\mathfrak{g}}_0 = \mathfrak{sp}(2n, \mathbb{C}) \oplus \mathbb{C}$$

canonically. The subalgebra $\mathfrak{sp}(2n, \mathbb{C})$ is formed by $\frac{1}{i\hbar}q(\hat{x}, \hat{\xi})$ where q are quadratic functions.

4.2. Fedosov connections. For any symplectic manifold M , we form a bundle of associative algebras $W = W_M$ and the bundles of Lie algebras $\mathfrak{g} = \mathfrak{g}_M$, $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_M$. We define them to be the bundles associated to the Sp -equivariant algebras W , \mathfrak{g} , etc. A *Fedosov connection* is an operator

$$\nabla : \Omega^\bullet(M, W) \rightarrow \Omega^{\bullet+1}(M, W)$$

such that:

(1)

$$\nabla = A_{-1} + \nabla_0 + A_1 + \dots$$

where $A_k \in \Omega^1(M, \mathfrak{g}_k)$ and ∇_0 is a connection in TM preserving ω ;

(2)

$$A_{-1} \in \Omega^1(M, \mathfrak{g}_{-1}) = \Omega^1(M, T_M^*)$$

is minus the map $T_M \rightarrow T_M^*$ defined by ω ;

(3) $\nabla^2 = 0$

A *lifting* of ∇ is an expression

$$\tilde{\nabla} = \tilde{A}_{-1} + \nabla_0 + \tilde{A}_0 + \tilde{A}_1 + \dots$$

where ∇_0 is the same connection which is now viewed as $\tilde{\mathfrak{g}}_0$ -valued, $\tilde{A}_k \in \Omega^1(M, \tilde{\mathfrak{g}}_k)$, and the image of \tilde{A}_k under the map induced by the projection $\tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ is A_k . (For the sake of uniformity we put $A_0 = 0$).

The following theorem is essentially due to Fedosov [Fe]. For expositions closer to ours, cf. [BNT], [NT1]. Another approach, which is valid for algebraic varieties, is contained in [BK].

Theorem 4.2.1. 1) For any Fedosov connection ∇ ,

$$\mathbb{A}_M = \mathbb{A}_M^\nabla = \ker(\nabla : C^\infty(M, W) \rightarrow \Omega^1(M, W))$$

is an algebra which is isomorphic to $C^\infty(M)[[\hbar]]$ as a $\mathbb{C}[[\hbar]]$ -module. The resulting product on $C^\infty(M)[[\hbar]]$ is a deformation quantization.

2) Any ∇ admits a lifting $\tilde{\nabla}$, and

$$\tilde{\nabla}^2 = \theta = \frac{1}{i\hbar}\omega + \sum_{k=0}^{\infty} (i\hbar)^k \theta_k \quad (4.8)$$

where θ_k are closed two-forms. For two different liftings of ∇ , the forms θ are cohomologous.

3) Deformation quantizations $\ker(\nabla)$ and $\ker(\nabla')$ are isomorphic if and only if the curvatures of their liftings are cohomologous.

4) Given two lifted Fedosov connections with the same curvature form θ , there is an isomorphism between $\ker(\nabla)$ and $\ker(\nabla')$ which is canonical up to a canonical inner isomorphism.

5) Any deformation quantization is isomorphic to $\ker(\nabla)$ for some ∇ .

6) For any closed θ such as in (4.8), there is a Fedosov connection ∇ with a lifting $\tilde{\nabla}$ such that $\tilde{\nabla}^2 = \theta$.

4.3. Groups of automorphisms of W and gauge transformations. Put $\tilde{\mathfrak{g}}_{\geq 1} = \prod_{k \geq 1} \tilde{\mathfrak{g}}_k$. This is a pronilpotent Lie algebra, so one can define

$$\tilde{G}_{\geq 1} = \exp \tilde{\mathfrak{g}}_{\geq 1}$$

Put also

$$\overline{G}_{\geq 0} = \mathrm{Sp}(2n, \mathbf{R}) \ltimes \tilde{G}_{\geq 1} \quad (4.9)$$

$$\tilde{G}_{\geq 0}^N = \widetilde{\mathrm{Sp}}^N(2n, \mathbf{R}) \ltimes \tilde{G}_{\geq 1} \quad (4.10)$$

Note that $\tilde{G}_{\geq 1}$ acts on $\tilde{\mathfrak{g}}$ -valued connections by gauge transformations. One can show that

Lemma 4.3.1. *Two Fedosov connections ∇, ∇' define isomorphic star products if and only if they are gauge equivalent, if and only if they have gauge equivalent liftings. Two lifted Fedosov connections are gauge equivalent if and only if their curvature forms are equal.*

This is the key part of the proof of the statements 3 and 4 of theorem 4.2.1. For example, to prove 4, observe that a gauge equivalence defines an isomorphism of corresponding bundles with connection, hence of the algebras of horizontal elements; two gauge equivalences of lifted connections differ by a gauge auto-equivalence, which is by definition an invertible section of the bundle W which is horizontal under ∇ , hence an invertible element of $\ker(\nabla)$.

4.3.1. *The Weil representation.* One can extend the Weil representation (cf. 2.2) as follows. Introduce a filtration on $\mathbb{C}[[\hat{x}^1, \dots, \hat{x}^n, \hbar, \hbar^{-1}]]$ which is multiplicative, \hat{x}^k are in F^1 , and \hbar in F^2 . Let

$$\widehat{V}^{\text{Weil}} = \bigoplus_T \exp\left(\frac{iT\hat{x}^2}{2\hbar}\right) \widehat{\mathbb{C}}[[\hat{x}^1, \dots, \hat{x}^n, \hbar, \hbar^{-1}]],$$

where the completion on the right is with respect to the filtration F and the summation is taken over all symmetric complex $n \times n$ matrices T with positive definite imaginary part. Let $\widehat{V}_0^{\text{Weil}}$ be a similar sum, but taken over all real $n \times n$ symmetric matrices. The action of $\widetilde{\text{Sp}}$ on V^{Weil} extends to an action of $\widetilde{G}_{\geq 0}$ on $\widehat{V}^{\text{Weil}}$. Similarly, the partial action of $\widetilde{\text{Sp}}$ on V_0^{Weil} extends to a partial action of $\widetilde{G}_{\geq 0}$ on $\widehat{V}_0^{\text{Weil}}$.

We treat $\widetilde{G}_{\geq 0}$ as a Lie group whose Lie algebra is $\widetilde{\mathfrak{g}}_{\geq 0}/\mathbb{C}$.

Lemma 4.3.2. *The subgroup P of $\widetilde{G}_{\geq 0}$ preserving the subspace $V_{T=0} = \mathbb{C}[[\hat{x}, \hbar]]$ is the Lie group of the Lie subalgebra $\{\frac{i}{\hbar}f | f \in \widehat{\xi}W_{\geq 1} + \hbar W_{\geq 1}\}$.*

The subgroup N of P of those elements whose action on the subspace $V_{T=0} = \mathbb{C}[[\hat{x}, \hbar]]$ is identity modulo \hbar is the Lie group of the Lie subalgebra $\{\frac{i}{\hbar}f | f \in \widehat{\xi}^2 W_{\geq 0} + \hbar \widehat{\xi} W_{\geq -1} + \hbar^2 W\}$.

4.4. **Fedosov construction and Lagrangian submanifolds.** Let L be a Lagrangian submanifold of a symplectic manifold M . We call a Fedosov connection ∇ compatible with L if the restriction of ∇ to L preserves the left ideal WT_L^\perp where $T_L^\perp \subset T_M^* \subset W$ is the annihilator of T_L .

The group of gauge transformations $\exp \widetilde{\mathfrak{g}}_{\geq 1}(L)$ acts on such connections, where

$$\widetilde{\mathfrak{g}}_{\geq 1}(L) = \{\sigma \in \Gamma(M, \widetilde{\mathfrak{g}}_{\geq 1}) : \sigma|_L \in \frac{1}{i\hbar} WT_L^\perp\} \quad (4.11)$$

The following is, essentially, a particular case of the statements contained in [Bo], [W].

Lemma 4.4.1. *The map sending ∇ to the cohomology class of $\widetilde{\nabla}^2$ defines a bijection between the set of gauge equivalence classes of Fedosov connections compatible with L and the affine set*

$$\frac{1}{i\hbar}[\omega] + H^2(M, L)[[\hbar]]$$

The proof goes exactly as for usual Fedosov connections.

4.5. The jet bundle J_M . For any manifold M of dimension m , let J be the bundle of jets of C^∞ functions. If $M = \cup U_\alpha$ is an open cover and a coordinate system x^1, \dots, x^m is chosen on every U_α , then we identify $J|_{U_\alpha}$ with $U_\alpha \times \mathbb{C}[[\hat{x}]]$ where we denote by $\mathbb{C}[[\hat{x}]]$ the algebra $\mathbb{C}[[\hat{x}^1, \dots, \hat{x}^m]]$. The transition functions of the bundle J are

$$G_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{Aut } \mathbb{C}[[\hat{x}]]$$

defined as follows:

$$G_{\alpha\beta}(x) : \hat{x} \mapsto g_{\alpha\beta}(\phi_\beta(x) + \hat{x}) - g_{\alpha\beta}(\phi_\beta(x)) \quad (4.12)$$

where

$$\phi_\alpha : U_\alpha \hookrightarrow \mathbb{R}^N$$

are the coordinate embeddings and

$$g_{\alpha\beta} = \phi_\alpha \phi_\beta^{-1}$$

The jet bundle is filtered by powers of the ideal $(\hat{x}^1, \dots, \hat{x}^m)$, and the associated graded bundle of algebras is $S[T_M^*]$, the symmetric algebra of the cotangent bundle. Using the fact that C_M^∞ is an acyclic sheaf, one shows that, noncanonically,

$$J_M \xrightarrow{\sim} S[[T_M^*]] \quad (4.13)$$

(the completion of $S[T_M^*]$) as bundles of algebras.

If a deformation quantization is given on M , then $J_M[[\hbar]]$ becomes a bundle of algebras. Locally, using the grading as in (4.5), put

$$F^k J[[\hbar]] = \prod_{p \geq k} J[[\hbar]]_p$$

The transition functions and the product preserve this filtration. The completed associated graded bundle of algebras is the Weyl bundle W . As above, one can show that

$$J_M[[\hbar]] \xrightarrow{\sim} W_M \quad (4.14)$$

as bundles of algebras.

For any M there is the canonical flat connection

$$\nabla_{\text{can}} : \Omega^\bullet(M, J) \rightarrow \Omega^{\bullet+1}(M, J) \quad (4.15)$$

which preserves the product. In coordinates,

$$(\nabla_{\text{can}} f)(x, \hat{x}) = \sum_{k=1}^m \left(\frac{\partial f}{\partial x^k} - \frac{\partial f}{\partial \hat{x}^k} \right) dx^k$$

The kernel of $\nabla_{\text{can}}|_{C^\infty(M, J)}$ is canonically isomorphic to the algebra $C^\infty(M)$.

If a deformation quantization is given on M , the image of ∇_{can} under the isomorphism (4.14) becomes a Fedosov connection. This proves the assertion 5) of theorem 4.2.1.

4.6. The groupoid \tilde{G} . This subsection is not used in the rest of the paper. We include it because we feel that its content might be useful in a modified theory of modules over deformation quantization algebras which is more suitable for applications.

Let M be a symplectic manifold with an $\widetilde{\text{Sp}}^N$ -structure. Let ∇ be a Fedosov connection with a lifting $\tilde{\nabla}$, and let $\mathbb{A}_M = \ker \nabla$ is the deformed sheaf of algebras of smooth functions. For two open subsets U and V of M , let

$$G_{UV} = \text{Iso}(\mathbb{A}_U, \mathbb{A}_V)$$

(the set of continuous isomorphisms of algebras). These sets form a groupoid whose objects are open subsets for M . In this subsection we define the groupoid \tilde{G}_{UV} together with a surjection

$$\tilde{G}_{UV} \rightarrow G_{UV} \tag{4.16}$$

such that there are central extensions

$$0 \rightarrow \mathbb{Z}/N \rightarrow \tilde{G}_{UU} \rightarrow G_{UU} \rightarrow 1 \tag{4.17}$$

Observe that $\text{Sp}(2n, \mathbf{R})$ acts on the group $\tilde{G}_{\geq 1}$ by conjugations, therefore we can construct a bundle of groups on M . If an $\widetilde{\text{Sp}}^N(2n, \mathbf{R})$ -structure on M is given, then one defines the induced groupoid $\tilde{G}_{\geq 0}^N$ with the manifold of objects M . Locally in coordinates, for $x, y \in M$, $\tilde{G}_{\geq 0}^N(x, y) = \tilde{G}_{\geq 0}^N$ (cf. (4.9)); the transition functions are left and right multiplications by the $\widetilde{\text{Sp}}^N$ -valued lifted transition functions of the tangent bundle T_M .

To define an element of \tilde{G}_{UV} , start with a symplectomorphism $f : V \xrightarrow{\sim} U$. Let $\tilde{g}(x) \in \tilde{G}_{\geq 0}^N(fx, x)$, $x \in V$, be a smooth family. We require the induced family $g(x) : W_{fx} \xrightarrow{\sim} W_x$ to preserve the Fedosov connection. By \tilde{G}_{UV} we denote the set of all such families $\tilde{g}(x)$.

This construction is an extension of a similar construction for symplectomorphisms which was defined in [Se].

4.7. Modified Fedosov construction. In this subsection, we observe that the Fedosov construction can be extended as follows. Recall that Fedosov's Weyl bundle W_M is the bundle whose fiber is the Weyl algebra W and whose transition functions are the images of the transition functions $g_{\alpha\beta}^T \in \text{Sp}(2n)$ in the group $G_{\geq 0}$ of automorphisms of W . A Fedosov connection is a flat connection of a special kind on this bundle.

To define its *lifting*, we used the fact that the transition functions of the bundle W admit a lifting to $\overline{G}_{\geq 0} = \mathrm{Sp}(2n, \mathbf{R}) \times \widetilde{G}_{\geq 1}$.

Now, let us start with any bundle of algebras \mathcal{W} whose transition functions take values in $G_{\geq 0}$:

$$G_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow G_{\geq 0}$$

We require that the projection of $G_{\alpha\beta}$ from $G_{\geq 0}$ to $\mathrm{Sp}(2n)$ coincide with $g_{\alpha\beta}^T$. A lifting of the transition functions is by definition a $\overline{G}_{\geq 0}$ -valued Čech one-cocycle $\widetilde{G}_{\alpha\beta}$ whose image under the projection $\overline{G}_{\geq 0} \rightarrow G_{\geq 0}$ is $G_{\alpha\beta}$.

By definition, a Fedosov connection in \mathcal{W} is a flat connection which preserves multiplication and whose A_{-1} term is as in 4.2. Given a lifting of the transition functions of \mathcal{W} , define a lifting of a Fedosov connection ∇ to be a $\widetilde{\mathfrak{g}}$ -valued connection $\widetilde{\nabla}$ whose image under the projection $\widetilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ is ∇ . More explicitly, it is a collection of forms

$$\widetilde{A}_\alpha = \sum_{k=-1}^{\infty} \widetilde{A}_{\alpha,k}; \quad \widetilde{A}_{\alpha,k} \in \Omega^1(U_\alpha, \widetilde{\mathfrak{g}}_k)$$

such that

$$\widetilde{A}_\alpha = \mathrm{Ad}(\widetilde{G}_{\alpha\beta})(\widetilde{A}_\beta) - \widetilde{G}_{\alpha\beta}^{-1} d\widetilde{G}_{\alpha\beta}$$

The following is a straightforward generalization of 4.2.1.

Theorem 4.7.1. 1) For any \mathcal{W} , there exist a Fedosov connection ∇ , a lifting $\widetilde{G}_{\alpha\beta}$ of the transition functions, and a lifting $\widetilde{\nabla}$ of ∇ . The algebra $\ker(\nabla : \Omega^0(M, \mathcal{W}) \rightarrow \Omega^1(M, \mathcal{W}))$ is isomorphic to a deformation quantization of (M, ω) .

2) For two bundles \mathcal{W} and \mathcal{W}' with lifted transition functions $\widetilde{G}_{\alpha\beta}$ and $\widetilde{G}'_{\alpha\beta}$ and for two lifted Fedosov connections $\widetilde{\nabla}$ and $\widetilde{\nabla}'$, if $\widetilde{\nabla}^2 = (\widetilde{\nabla}')^2$ then the algebras $\ker(\nabla)$ and $\ker(\nabla')$ are isomorphic. Moreover, the isomorphism is canonical up to a canonical inner automorphism.

3) The curvature form

$$\theta = \frac{1}{i\hbar} \omega + \sum_{k=0}^{\infty} (i\hbar)^k \theta_k, \quad \theta_k \in \Omega^2(M),$$

is closed. Its cohomology class is the complete invariant of the deformation up to isomorphism.

Remark 4.7.2. In the construction above, one can take $G_{\alpha\beta}$ to be the transition functions of the bundle of jets, and ∇ to be the canonical connection ∇_{can} . We will see an example of this in 4.9.

4.8. The canonical stack on a symplectic manifold. We can strengthen the statement of Theorem 4.7.1 as follows.

Proposition 4.8.1. *1) For two bundles \mathcal{W} and \mathcal{W}' with lifted transition functions $\tilde{G}_{\alpha\beta}$ and $\tilde{G}'_{\alpha\beta}$ and for two lifted Fedosov connections $\tilde{\nabla}$ and $\tilde{\nabla}'$, if $\tilde{\nabla}^2 = (\tilde{\nabla}')^2$ then there is a canonical isomorphism of algebras $G(\tilde{\nabla}, \tilde{\nabla}') : \ker(\tilde{\nabla}') \rightarrow \ker(\tilde{\nabla})$.*

2) For three bundles \mathcal{W} and \mathcal{W}' with lifted transition functions $\tilde{G}_{\alpha\beta}$, $\tilde{G}'_{\alpha\beta}$, $\tilde{G}''_{\alpha\beta}$, and for three lifted Fedosov connections $\tilde{\nabla}$, $\tilde{\nabla}'$, $\tilde{\nabla}''$, if $\tilde{\nabla}^2 = (\tilde{\nabla}')^2 = (\tilde{\nabla}'')^2$ then there is a canonical element $c(\tilde{\nabla}, \tilde{\nabla}', \tilde{\nabla}'')$ of $\ker(\tilde{\nabla})$ which is congruent to 1 modulo \hbar , such that

$$G(\tilde{\nabla}, \tilde{\nabla}')G(\tilde{\nabla}', \tilde{\nabla}'') = \text{Ad}(c(\tilde{\nabla}, \tilde{\nabla}', \tilde{\nabla}''))G(\tilde{\nabla}, \tilde{\nabla}'')$$

3)

$$c(\tilde{\nabla}, \tilde{\nabla}', \tilde{\nabla}'')c(\tilde{\nabla}, \tilde{\nabla}'', \tilde{\nabla}''') = G(\tilde{\nabla}, \tilde{\nabla}')(c(\tilde{\nabla}', \tilde{\nabla}'', \tilde{\nabla}'''))c(\tilde{\nabla}, \tilde{\nabla}', \tilde{\nabla}''')$$

This provides a canonical stack of deformation quantizations on every symplectic manifold, as well as on any symplectic manifold with a pseudogroup of symplectomorphisms. Cf. [Kas] and [PS] for a more analytical construction which uses microdifferential operators, as well [DP] for some further discussion and applications.

Proof of the Proposition For any $\tilde{\nabla}$ and $\tilde{\nabla}'$ with the same curvature form, there exists a gauge transformation $\sigma(\tilde{\nabla}, \tilde{\nabla}')$ between $\tilde{\nabla}'$ and $\tilde{\nabla}$. Let $G(\tilde{\nabla}, \tilde{\nabla}')$ be the action of this gauge transformation reduced to horizontal sections. Put also

$$c(\tilde{\nabla}, \tilde{\nabla}', \tilde{\nabla}'') = \sigma(\tilde{\nabla}, \tilde{\nabla}')\sigma(\tilde{\nabla}', \tilde{\nabla}'')\sigma(\tilde{\nabla}, \tilde{\nabla}'')^{-1}$$

It is easy to see that these G and c satisfy all the properties stated in the Proposition.

4.9. Differential operators and the deformation quantization of $T^*(X)$. Let X be a manifold. For the sheaf of rings $D_X^{\frac{1}{2}}$ of differential operators on half-densities on X , let $F_p D_X^{\frac{1}{2}}$ be the filtration by order. Let

$$\mathcal{R}D_X^{\frac{1}{2}} = \bigoplus_{p \geq 0} \hbar^p F_p D_X^{\frac{1}{2}} \quad (4.18)$$

be the Rees ring. Let $X = \cup_{\alpha} U_{\alpha}^0$ be an open cover. A choice of coordinates on U_{α}^0 identifies $\mathcal{R}D_{U_{\alpha}^0}^{\frac{1}{2}}$ with the ring $C_{U_{\alpha}^0}^{\infty}[\xi^1, \dots, \xi^n, \hbar]$ where

$$\xi^k = i\hbar \frac{\partial}{\partial x^k}$$

The latter ring can be identified with the ring $C_{\pi^{-1}U_\alpha^0}^{\infty, \text{poly}}$ where $\pi : T^*X \rightarrow X$ is the projection and $C^{\infty, \text{poly}}$ stands for the sheaf of C^∞ functions which are polynomial along the fibers. We use the Weyl identification, analogous to one that was used in the discussion after (4.4).

We get

$$\phi_\alpha : \mathcal{R}D_{U_\alpha^0}^{\frac{1}{2}} \xrightarrow{\sim} C_{\pi^{-1}U_\alpha^0}^{\infty, \text{poly}}$$

One checks that the product on $\mathcal{R}D_{U_\alpha^0}^{\frac{1}{2}}$ induces on the right hand side a star product which we denote by $*_\alpha$. This star product extends to $C_{\pi^{-1}U_\alpha^0}^\infty$. Furthermore, $G_{\alpha\beta} = \phi_\alpha \phi_\beta^{-1}$ extend to isomorphisms between $*_\alpha$ and $*_\beta$ in the sense of definition 4.1.1. Using partitions of unity, one constructs automorphisms T_α of $*_\alpha$ such that $G_{\alpha\beta} = T_\alpha T_\beta^{-1}$. This allows to define a star product on T^*X .

Let us recall how one identifies the above deformation in terms of the classification theorem 4.2.1.

Proposition 4.9.1. *The characteristic class θ of this deformation is $\frac{1}{i\hbar}\omega$ ($= 0$).*

Proof. The statement itself is straightforward. Indeed, one checks that our deformation is isomorphic to its opposite, and it is easy to see that the characteristic class of the opposite is minus the original characteristic class. We will need, however, an explicit description of our deformation in terms of Section 4.7.

Start with an open cover $\{\pi^{-1}(U_\alpha^0)\}$ of $T^*(X)$. A choice of coordinates $x_\alpha = (x_\alpha^1, \dots, x_\alpha^n)$ on U_α^0 determines a coordinate system (x_α, ξ^α) on $\pi^{-1}(U_\alpha^0)$. The transition functions between two different coordinate systems are

$$\begin{aligned} x_\alpha &= g_{\alpha\beta}(x_\beta) \\ \xi^\alpha &= {}^t g'_{\alpha\beta}(x_\beta)^{-1} \xi^\beta \end{aligned} \tag{4.19}$$

(${}^t g'_{\alpha\beta}(x_\beta)$ stands for the transposed Jacobi matrix).

Now consider the bundle $\pi^* \text{jets } \mathcal{R}D_X^{\frac{1}{2}}$ with the canonical connection $\pi^* \nabla_{\text{can}}$. (To construct the bundle of jets one acts as in 4.5). This is a bundle of algebras with the fiber

$$W_{\text{fin}} = \mathbb{C}[[\widehat{x}]][\widehat{\xi}, \hbar] \tag{4.20}$$

(again, we use the Weyl identification of the two sides). Its transition functions are given explicitly as follows:

$$\widehat{x} \mapsto g_{\alpha\beta}(x_\beta + \widehat{x}) - x_\alpha \tag{4.21}$$

$$\widehat{\xi} \mapsto {}^t g'_{\alpha\beta}(x_\beta + \widehat{x})^{-1} \widehat{\xi} \quad (4.22)$$

Because of the presence of the half-densities, the product in the second equation is the commutative product, not the composition in the Weyl algebra (as always, we use the Weyl identification between functions and operators). The canonical connection is, in coordinates, given by

$$\pi^* \nabla_{\text{can}} = d - \sum \frac{\partial}{\partial \widehat{x}^k} dx^k,$$

without a similar ξ term. To correct that, apply the gauge transformation

$$\sigma_\alpha = \exp \text{ad} \frac{1}{i\hbar} \xi^\alpha \widehat{x} \in \text{Aut}(W_{\text{fin}}) \quad (4.23)$$

We get a new bundle of algebras whose transition functions are

$$\widehat{x} \mapsto g_{\alpha\beta}(x_\beta + \widehat{x}) - x_\alpha \quad (4.24)$$

$$\widehat{\xi} \mapsto {}^t g'_{\alpha\beta}(x_\beta + \widehat{x})^{-1} (\xi^\beta + \widehat{\xi}) - \xi^\alpha \quad (4.25)$$

Again, the multiplication in the second formula is the commutative multiplication of power series. Therefore, these transition functions coincide with the transition functions of the jet bundle J_{T^*X} (compare with (4.19)). Note also that these transition functions admit a canonical lifting to $\overline{G}_{\geq 0}$ and even to $\widetilde{G}_{\geq 0}$. Indeed, consider the group K of formal automorphisms of the trivial line bundle on \mathbb{R}^n acting by

$$f(\widehat{x}) \mapsto p(\widehat{x}) f(g(\widehat{x})) |\det g'(\widehat{x})|^{\frac{1}{2}} \quad (4.26)$$

where g is a formal diffeomorphism of the form $g : \widehat{x} \mapsto a\widehat{x} + o(\widehat{x})$, $a \in \text{GL}(n, \mathbb{R})$, and $p(\widehat{x}) \in \mathbb{C}[[\widehat{x}]]$, $p(0) = 0$. The group K maps into $G_{\geq 0}$ as follows. We can represent W as an algebra of operators on $\mathbb{C}[[\widehat{x}, \hbar]]$ by identifying, as above, $\widehat{x}^n \widehat{\xi}^m$ with the symmetrized product $\widehat{x}^n (i\hbar \frac{\partial}{\partial \widehat{x}})^m$. Then K acts on these operators by conjugation. To check that there is a canonical lifting $K \rightarrow \widetilde{G}_{\geq 0}$, observe that

$$K = \text{GL}(n, \mathbb{R}) \ltimes K_{\geq 1}$$

where $K_{\geq 1}$ is the group of elements for which $a = 1$. But GL maps to Sp , and this map lifts to $\widetilde{\text{Sp}}$; on the other hand, $K_{\geq 1}$ has a canonical lifting to $\widetilde{G}_{\geq 1}$. Indeed,

$$K_{\geq 1} = \exp(\mathfrak{k}_{\geq 1})$$

where

$$\mathfrak{k}_{\geq 1} = \left\{ \frac{1}{i\hbar} P(\widehat{x}) \widehat{\xi} + Q(\widehat{x}) \mid P(\widehat{x}) = o(\widehat{x}); Q(0) = 0 \right\}$$

and this pronilpotent Lie algebra is a subalgebra of $\widetilde{\mathfrak{g}}_{\geq 1}$, not just of $\mathfrak{g}_{\geq 1}$.

As for the connection, $\pi^*\nabla_{\text{can}}$ becomes a Fedosov connection after the gauge transformation σ_α . It has a flat lifting which has an extra summand $\frac{i}{\hbar}\xi dx$. Subtract it, and we get a Fedosov connection, together with a lifting whose curvature is $\frac{1}{i\hbar}\omega$.

Therefore we are exactly in the situation of Theorem 4.7.1 and Remark 4.7.2. This proves the proposition.

5. PRELIMINARIES FROM LAGRANGIAN ANALYSIS

5.1. Let us recall the classical construction from [H], [GS] in terms that are suited for our purposes. Let X be a manifold. Consider an open coordinate cover $X = \cup U_\alpha^0$, and a refinement of the cover $\pi^{-1}U_\alpha$: $T^*X = \cup U_\beta$; $U_\beta \subset \pi^{-1}U_{\alpha(\beta)}^0$. Let $L \subset T^*X$ be a Lagrangian submanifold. Note that at this stage we do not assume any of the subsets to be conical. Denote by \mathcal{E}_L the local system with the transition functions $\exp(\frac{i\pi}{2}c)$ where c is a \mathbb{Z} -valued one-cocycle representing the Maslov class of L .

1) For any section

$$a \in \Gamma_c(U_\beta, |\Omega_L|^{\frac{1}{2}} \otimes \mathcal{E}_L)$$

and any $\hbar \neq 0$, one can construct a half-density

$$\widehat{a}_{\beta, \hbar} \in \Gamma(U_{\alpha(\beta)}^0, |\Omega_X|^{\frac{1}{2}}) \quad (5.1)$$

2) For any smooth function f on $\pi^{-1}(U_\alpha^0)$, polynomial along the fibers, and any $\hbar \neq 0$, one can construct a differential operator $\widehat{f}_{\alpha, \hbar}$ on half-densities on U_α^0 .

3) If a is supported in $U_\beta \cap U_\gamma$ then

$$\widehat{a}_{\gamma, \hbar} = \widehat{a}_{\beta, \hbar} + \sum_{k=1}^{\infty} (i\hbar)^k \widehat{R_{\beta, \gamma, k}}(a)_{\beta, \hbar} + O(\hbar^\infty)$$

as $\hbar \rightarrow 0$, where $R_{\beta, \gamma, k}$ are differential operators on sections of $|\Omega_L|^{\frac{1}{2}} \otimes \mathcal{E}_L$.

4) If f is supported in $U_\alpha^0 \cap U_{\alpha_1}^0$, then

$$\widehat{f}_{\alpha_1, \hbar} = \widehat{f}_{\alpha, \hbar} + \sum_{k=1}^{\infty} (i\hbar)^k \widehat{T_{\alpha, \alpha_1, k}}(f)_{\alpha, \hbar} + O(\hbar^\infty)$$

where $T_{\alpha, \alpha_1, k}$ are differential operators.

5)

$$\widehat{f}_{\alpha(\beta),\hbar}(\widehat{a}_{\beta,\hbar}) = (f|_L \cdot a)_{\beta,\hbar} + \sum_{k=1}^{\infty} (i\hbar)^k \widehat{P_{\beta,k}(f,a)}_{\beta,\hbar} + O(\hbar^\infty)$$

where $P_{\beta,k}$ are bidifferential expressions depending on a and on the jet of f at L .

6)

$$\widehat{f}_{1\alpha,\hbar} \circ \widehat{f}_{2\alpha,\hbar} = \widehat{f_1 \cdot f_{2\alpha,\hbar}} + \sum_{k=1}^{\infty} (i\hbar)^k \widehat{D_{\alpha,k}(f_1, f_2)}_{\alpha,\hbar} + O(\hbar^\infty)$$

where $D_{\alpha,k}$ are bidifferential expressions.

We see that the asymptotic expressions 3) - 6) define a deformation quantization of T^*X and a sheaf of modules V_L^H over the deformed algebra $\mathbb{A}_{T^*X} = C_{T^*X}^\infty[[\hbar]]$, supported on L .

Let us briefly recall how to construct $\widehat{a}_{\beta,\hbar}$. Locally on U_β , consider a *phase function* of L : if $x = (x^1, \dots, x^n)$ are coordinates on $U_{\alpha(\beta)} \subset X$, let θ be a variable in \mathbb{R}^k ; a phase function is a function $\varphi(x, \theta)$ such that:

i)

$$L = \{(x, \xi) | d_\theta \varphi(x, \theta) = 0; \xi = d_x \varphi(x, \theta)\}$$

ii) The Hessian $(\varphi_{x\theta}, \varphi_{\theta\theta})$ is of maximum rank k .

Given a phase function, the map

$$\begin{aligned} i : \{(x, \theta) | d_\theta \varphi(x, \theta) = 0\} &\rightarrow L; \\ (x, \theta) &\mapsto (x, d_x \varphi(x, \theta)) \end{aligned}$$

is a local diffeomorphism. The left hand side is a submanifold in \mathbb{R}^{n+k} .

The function $\varphi(i^{-1}x)$, which we still denote by φ , satisfies $d\varphi = \xi dx$, the right hand side being the canonical one-form on T^*X (its differential is ω , so it is closed on L).

The Hörmander construction is as follows. Start with a local section a on U_β . Given a phase function $\varphi = \varphi_\beta$ on U_β , and given local coordinates on $U_{\alpha(\beta)}$, represent, locally, a as a function on $\{(x, \theta) | d_\theta \varphi(x, \theta) = 0\}$. Extend it to a function of x, θ which is zero away from a small neighborhood of $i^{-1}L$ as follows. Subdivide the $n + k$ variables (x, θ) into two groups $y \in \mathbb{R}^n$, $z \in \mathbb{R}^k$, such that the Hessian $k \times k$ matrix $\partial_{z\theta}^2 \varphi$ is nondegenerate. Observe that the restriction of the map (6.4) to the subspace $z = 0$ is a local diffeomorphism with L ; extend a to a function in y only, and multiply by a function in y which is zero away from a neighborhood of the origin. Now define

$$\widehat{a}_{\beta,\hbar} = \left[\int e^{\frac{i}{\hbar} \varphi_\beta(x, \theta)} a(x, \theta) d\theta \right] |dx|^{\frac{1}{2}} \quad (5.2)$$

One can now proceed to generalize this to the case when L is a homogeneous Lagrangian submanifold, φ is homogeneous of degree one in θ , and $a(x, \theta)$ satisfying certain standard growth conditions with respect to θ . One can still define $\widehat{a}_{\beta, \hbar}$ as distributional half-densities, whose action on a test half-density $u(x)|dx|^{\frac{1}{2}}$ is defined as

$$\widehat{a}_{\beta, \hbar}(u) = \int e^{\frac{i}{\hbar}\varphi_{\beta}(x, \theta)} a(x, \theta) u(x) d\theta dx$$

The latter integral is taken using the stationary phase method as explained in [GS] and [H].

6. LAGRANGIAN ANALYSIS AND DEFORMATION QUANTIZATION

6.1. Let $L \subset T^*X$ be a Lagrangian submanifold. As above, let $X = \cup_{\alpha} U_{\alpha}^0$ and $T^*X = \cup_{\beta} U_{\beta}$, a cover which is a refinement of $\{\pi^{-1}U_{\alpha}^0\}$. Let φ_{β} be a phase function of $L|_{U_{\beta}}$. Using these data, we will construct a sheaf of modules over $\mathbb{A}_{T^*X}^{\hbar} \otimes_{\mathbb{C}[[\hbar]]} \mathbb{K}$ where $\mathbb{A}_{T^*X}^{\hbar}$ is the deformation quantization discussed in 4.9 and $\mathbb{K} = \mathbb{C}[[\hbar]][\hbar^{-1}, e^{\frac{i}{\hbar}a} | a \in \mathbb{R}]$.

Denote by V_{β} the space of formal expressions

$$\frac{1}{(2\pi\hbar)^{k/2}} \left[\int e^{\frac{i}{\hbar}\varphi(x, \theta)} a(x, \theta) d\theta \right] |dx|^{\frac{1}{2}} \quad (6.1)$$

where $a(x, \theta)$ is a $\mathbb{C}[[\hbar]]$ -valued smooth function and k is the dimension of the θ space (we will mostly use local phase functions of special kind for which $k = n$). More precisely,

$$V_{\beta} = \{a(x, \theta)\} / \sim \quad (6.2)$$

where

$$\varphi_{\theta} \cdot a \sim i\hbar a_{\theta} \quad (6.3)$$

and $a(x, \theta)$ is a $\mathbb{C}[[\hbar]]$ -valued smooth function on the preimage of U_{β} under the map

$$(x, \theta) \mapsto (x, d_x\varphi(x, \theta)) \quad (6.4)$$

Note that V_{β} as a $\mathbb{C}[[\hbar]]$ -module is isomorphic to $C^{\infty}(L \cap U_{\beta})[[\hbar]]$. Indeed, using the discussion before the formula (5.2), we see that the map

$$a(y) \mapsto (a \bmod \sim) \in V_{\beta}$$

is an isomorphism.

Now we have to define the transition functions

$$G_{\beta\gamma} : V_{\beta}|_{U_{\beta} \cap U_{\gamma}} \xrightarrow{\sim} V_{\gamma}|_{U_{\beta} \cap U_{\gamma}}$$

and the action of $\pi^* \mathcal{R}D_{U_\alpha^0}^{\frac{1}{2}}$ on V_β where $\alpha = \alpha(\beta)$. Both are suggested by (6.1). The action is defined by

$$\begin{aligned} x^m \cdot a &= x^m a \\ \xi^m \cdot a &= i\hbar \frac{\partial a}{\partial x^m} - \frac{\partial \varphi}{\partial x^m} \cdot a \end{aligned} \quad (6.5)$$

for $m = 1, \dots, n$. This action preserves the equivalence relation (6.3).

As for the transition functions, let us start, following Hörmander, by introducing coordinate changes

$$\varphi(x, \theta) \mapsto \varphi(g(x), \rho(x, \theta)) \quad (6.6)$$

where g is a local diffeomorphism. It is straightforward that a phase function obtained by a coordinate change from φ defines the same Lagrangian submanifold as φ . Hörmander proved that, locally on U_β , any two phase functions differ by such a coordinate change, followed by addition of $\pm \sum_{i=1}^c \theta_i^2/2$. As we will see later, the numbers $\pm c = \mu_{\beta\gamma}$ define a cocycle representing twice the Maslov class).

Let us write down the transition functions corresponding to the change (6.6) where

$$g = g_{\beta\gamma} = g_{\alpha(\beta)\alpha(\gamma)}$$

are the transition functions of the manifold X and $\rho = \rho_{\beta\gamma}$. They act on the equivalence classes of formal expressions (6.1) as follows:

$$\begin{aligned} G_{\beta\gamma} : \frac{1}{(2\pi\hbar)^{k/2}} \int e^{\frac{i}{\hbar}\varphi(x,\theta)} a(x_\beta, \theta) d\theta |dx_\beta|^{\frac{1}{2}} \mapsto \\ \frac{1}{(2\pi\hbar)^{k/2}} \int e^{\frac{i}{\hbar}\varphi(g_{\beta\gamma}(x_\gamma), \rho_{\beta\gamma}(x_\gamma, \theta))} a(g_{\beta\gamma}(x_\gamma), \rho_{\beta\gamma}(x_\gamma, \theta)) \times \\ \times |\det \frac{\partial \rho_{\beta\gamma}(x_\gamma, \theta)}{\partial \theta}|^{-1} |\det g'_{\beta\gamma}(x_\gamma)|^{\frac{1}{2}} d\theta |dx_\gamma|^{\frac{1}{2}} \end{aligned} \quad (6.7)$$

Each summand $\pm \theta_i^2$ contributes a multiple

$$\frac{1}{(2\pi\hbar)^{1/2}} \int e^{\pm \frac{i\theta^2}{2\hbar}} d\theta = e^{\mp \frac{i\pi}{4}}$$

To show that $G_{\beta\gamma} G_{\gamma\delta} = G_{\beta\delta}$, note that, though two phase functions may differ from one another by more than one coordinate change, two different coordinate changes define the same transformation of the space $\{a(x, \theta)\}/\sim$, provided that the underlying changes of the coordinate x are the same.

Example 6.1.1. Let L be given by the equation $\xi = kx$ in \mathbb{R}^2 . Assume that $k \neq 0$. Consider two phase functions of $x = x_\beta = x_\gamma$;

$$\varphi_\beta(x) = k \frac{x^2}{2};$$

$$\varphi_\gamma(x, \theta) = x\theta - k^{-1} \frac{\theta^2}{2}$$

The phase function φ_γ can be obtained from $\varphi_\beta - \text{sgn}(k) \frac{\theta^2}{2}$ by a coordinate change

$$g_{\beta\gamma}(x) = x;$$

$$\rho_{\beta\gamma}(x, \theta) = \sqrt{|k|x} - \frac{\text{sgn}(k)}{\sqrt{|k|}} \theta$$

The transition functions act as follows:

$$\begin{aligned} e^{\frac{i}{\hbar} \frac{kx^2}{2}} a(x) |dx|^{\frac{1}{2}} &\mapsto \\ \frac{1}{(2\pi)^{\frac{1}{2}}} e^{\text{sgn}(k) \frac{\pi i}{4}} \int e^{\frac{i}{\hbar} (\frac{kx^2}{2} - \text{sgn}(k) \frac{\theta^2}{2})} a(x) d\theta |dx|^{\frac{1}{2}} &\mapsto \end{aligned} \quad (6.8)$$

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \sqrt{|k|} e^{\text{sgn}(k) \frac{\pi i}{4}} \int e^{\frac{i}{\hbar} (x\theta - \frac{\theta^2}{2k})} a(x) d\theta |dx|^{\frac{1}{2}} \quad (6.9)$$

In other words, an element of V_β is a function of x and \hbar . An element of V_γ is a function of x, θ , and \hbar , modulo equivalence

$$i\hbar \partial_\theta a \sim (x - k^{-1}\theta) a$$

In every equivalence class there is unique function depending on θ and \hbar and not on x . The transition function acts by taking $a(x, \hbar)$, multiplying it by $\sqrt{|k|} e^{\text{sgn}(k) \frac{\pi i}{4}}$, and then rewriting it as a function of θ and \hbar using the above equivalence relation.

Let us introduce a special class of phase functions generalizing the above example, cf. [GS]. For any β , choose coordinates on $U_{\alpha(\beta)}$ such that $x = (x_1, x_2)$ where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$, such that L can be defined by equations

$$\begin{aligned} \xi_1 &= F_{x_1}(x_1, \xi_2) \\ x_2 &= -F_{\xi_2}(x_1, \xi_2) \end{aligned} \quad (6.10)$$

where ξ_1, ξ_2 are coordinates dual to x_1, x_2 . This is equivalent to the requirement that the projection of L to $\{\xi_1 = x_2 = 0\}$ along $\{\xi_2 = x_1 = 0\}$ is an isomorphism on U_β .

Now, put $\theta = (\xi_1, \xi_2)$ and

$$\varphi_\beta(x, \theta) = \varphi(x, \theta) = x_2 \xi_2 + F(x_1, \xi_2) + \frac{1}{2}(\xi_1 - F_{x_1})^2 \quad (6.11)$$

This is a local phase function for L .

For the above phase functions, it is easy to see that V_β as a $\mathbb{C}[[\hbar]]$ -module is isomorphic to $C^\infty(L \cap U_\beta)[[\hbar]]$. Indeed, observe that the restriction of the above map to $\{x_1, 0, 0, \xi_2\}$ is a local diffeomorphism with L ; on the other hand, the map $a(x_1, \xi_2) \mapsto (a \bmod \sim) \in V_\beta$ is an isomorphism.

More precisely, the equivalence relation for a is:

$$\begin{aligned} i\hbar \frac{\partial a}{\partial \xi_1} &\sim (\xi_1 - F_{x_1})a \\ i\hbar \frac{\partial a}{\partial \xi_2} &\sim (x_2 + F_{\xi_2})a \end{aligned} \quad (6.12)$$

This equivalence allows to identify elements of V_β with functions of x_1, ξ_2, \hbar . Under this identification, the algebra $\mathbb{A}_{U_\beta}^\hbar$ acts by

$$\begin{aligned} x_1 &\mapsto x_1 \\ x_2 &\mapsto i\hbar \frac{\partial}{\partial \xi_2} - \frac{\partial F}{\partial \xi_2} \\ \xi_1 &\mapsto i\hbar \frac{\partial}{\partial x_1} + \frac{\partial F}{\partial x_1} \\ \xi_2 &\mapsto -\xi_2 \end{aligned} \quad (6.13)$$

In other words, locally,

$$\mathbb{A}_{U_\beta}^\hbar / I_F \quad (6.14)$$

where I_F is the left ideal generated by the local equations $x_2 + F_{\xi_2}, \xi_1 - F_{x_1}$ of L .

Now let us describe the transition functions for this special choice of the phase functions. This will generalize Example 6.1.1.

Assume that on U_β L is presented as

$$\begin{aligned} \xi_1 &= F_{x_1}(x_1, x_2, \xi_3, \xi_4) \\ \xi_2 &= F_{x_2}(x_1, x_2, \xi_3, \xi_4) \\ x_3 &= -F_{\xi_3}(x_1, x_2, \xi_3, \xi_4) \\ x_4 &= -F_{\xi_4}(x_1, x_2, \xi_3, \xi_4) \end{aligned} \quad (6.15)$$

$$x_4 = -F_{\xi_4}(x_1, x_2, \xi_3, \xi_4) \quad (6.16)$$

and on U_γ

$$\begin{aligned}\xi_1 &= G_{x_1}(x_1, x_2, \xi_3, \xi_4) \\ x_2 &= -G_{\xi_2}(x_1, x_2, \xi_3, \xi_4) \\ \xi_3 &= G_{x_3}(x_1, x_2, \xi_3, \xi_4) \\ x_4 &= -G_{\xi_4}(x_1, x_2, \xi_3, \xi_4)\end{aligned}\tag{6.17}$$

$$\tag{6.18}$$

This means that the Hessian matrix $\text{Hess}_{\xi_2, x_3}(F)$ is nondegenerate. If $a_\gamma = G_{\beta\gamma}(a_\beta)$, then the following two expressions should be the same:

$$\begin{aligned}\int e^{\frac{i}{\hbar}(\xi_3 x_3 + \xi_4 x_4 + F + \frac{1}{2}(\xi_1 - F_{x_1})^2 + \frac{1}{2}(\xi_2 - F_{x_2})^2)} a_\beta(x_1, x_2, \xi_3, \xi_4) d\xi_1 d\xi_2 d\xi_3 d\xi_4 = \\ \int e^{\frac{i}{\hbar}(\xi_2 x_2 + \xi_4 x_4 + G + \frac{1}{2}(\xi_1 - G_{x_1})^2 + \frac{1}{2}(\xi_3 - G_{x_3})^2)} a_\gamma(x_1, x_3, \xi_2, \xi_4) d\xi_1 d\xi_2 d\xi_3 d\xi_4\end{aligned}$$

Here, as above, we use the rule

$$\frac{1}{\sqrt{2\pi\hbar}} \int e^{\frac{i}{\hbar} k x^2 / 2} dx = \frac{1}{\sqrt{ik}}, \tag{6.19}$$

$k \neq 0$, where we choose the branch of the square root

$$\sqrt{re^{i\varphi}} = \sqrt{r}e^{i\varphi/2}, \quad \varphi \neq \pi$$

Thus,

$$e^{\frac{i}{\hbar}G} a_\gamma = e^{\frac{\pi i}{4}(n_3 - n_2)} \text{Fourier}_{\xi_3 \rightarrow x_3; x_2 \rightarrow -\xi_2}(e^{\frac{i}{\hbar}F} a_\beta) \tag{6.20}$$

Here, for $x \in \mathbb{R}^N$,

$$(\text{Fourier } f)(\xi) = \frac{1}{(2\pi\hbar)^{N/2}} \int e^{\frac{i}{\hbar}x\xi} f(x) dx \tag{6.21}$$

It remains to make sense of an expression

$$\text{Fourier } e^{\frac{i}{\hbar}F(y)} a(y) \tag{6.22}$$

where F is a smooth function with an isolated critical point y_0 at which the Hessian is nondegenerate. We define this via a well known asymptotic expansion

$$\text{Fourier } e^{\frac{i}{\hbar}F(y)} a(y) = e^{\frac{i}{\hbar}G(\eta)} \exp\left(\frac{\pi i}{4} \text{sgn Hess}_{y_0} F\right) |\text{Hess}_{y_0} F|^{-\frac{1}{2}} \sum_{k=0}^{\infty} b_k(\eta) \hbar^k \tag{6.23}$$

Here $G(\eta)$ is defined by

$$G(\eta) = \eta y + F(y),$$

y being the solution of

$$\eta + F'(y) = 0$$

(the Legendre transform of F).

Remark 6.1.2. Let us stress that the above expansion makes sense for a power series F with nondegenerate Hessian and for a power series $a(y)$. Then G and b are also power series. For the Fourier transform this was already explained in 2.2 and 4.3.1.

For open subsets U_β and U_γ put

$$c_{\beta\gamma} = \frac{1}{2} \text{signature Hess}_{\xi_2, x_3}(F) \quad (6.24)$$

By theorem 3.1.1, the cochain c is a $\frac{1}{2}\mathbb{Z}$ -valued Čech one-cocycle of L which is cohomologous to a \mathbb{Z} -valued cocycle representing the Maslov class of L .

Let $\alpha = \xi dx$ be the canonical one-form; since $d\alpha = \omega$, $\alpha|_L$ is closed. The choice of local phase functions allows us to represent it by a Čech one-cocycle α_L with values in \mathbb{R} :

$$\alpha_{\beta\gamma} = \varphi_\beta - \varphi_\gamma \quad (6.25)$$

on L . The right hand side is locally constant since for all β $d\varphi_\beta = \alpha$ on L . We have proven the following statement.

Proposition 6.1.3. *The sheaf V defined via local modules V_β and transition functions $G_{\beta\gamma}$ is a sheaf of $\mathbb{A}^h \otimes_{\mathbb{C}[[\hbar]]} \mathbb{K}$ -modules supported on L . Locally, $V \xrightarrow{\sim} |\Omega_L|^{\frac{1}{2}} \otimes_{\mathbb{C}} \mathbb{K}$, and the transition functions are of the form*

$$G_{\beta\gamma} = e^{\frac{i}{\hbar}\alpha_L + \frac{\pi i}{2}\mu_L} g_{\beta\gamma}(\hbar)$$

where μ_L is a \mathbb{Z} -valued cocycle defining the Maslov class of L and $g_{\beta\gamma} = 1 \pmod{\hbar}$

Definition 6.1.4. *We denote the above sheaf of modules by V_L .*

7. THE LAGRANGIAN JET BUNDLE

7.1. Next we observe that V_L is the sheaf of horizontal sections of a module over the algebra of jets of functions on T^*X with the star product constructed above. This module of jets will be equipped with a flat connection compatible with the canonical connection in the bundle of jet algebras.

First, define, for an open subset U_β and a phase function $\varphi = \varphi_\beta$,

$$J_\beta = \{e^{\frac{i}{\hbar}\varphi_\beta(x+\widehat{x}, \theta+\widehat{\theta})} a(x, \theta; \widehat{x}, \widehat{\theta}; \hbar) |dx|^{\frac{1}{2}}\} / \sim \quad (7.1)$$

where (x, θ) are in the preimage of $L \cap U_\beta$ under the map $(x, \theta) \mapsto (x, d_x \varphi(x, \theta))$ and

$$e^{\frac{i}{\hbar} \varphi(x+\hat{x}, \theta+\hat{\theta})} \varphi_\theta(x+\hat{x}, \theta+\hat{\theta}) a |dx|^{\frac{1}{2}} \sim i \hbar e^{\frac{i}{\hbar} \varphi(x+\hat{x}, \theta+\hat{\theta})} a_{\hat{\theta}} |dx|^{\frac{1}{2}} \quad (7.2)$$

Locally, J_β is isomorphic to the space of sections on U_β of the bundle of jets of half-densities on L ($\mathbb{C}[[\hbar]]$ -valued).

One defines the transition functions $J_\beta \xrightarrow{\sim} J_\gamma$ on $U_\beta \cap U_\gamma$ and the action of π^* jets $\mathcal{R}D_X^{\frac{1}{2}}$. The latter is defined by

$$\begin{aligned} \widehat{\xi}^m : e^{\frac{i}{\hbar} \varphi} a |dx|^{\frac{1}{2}} &\mapsto e^{\frac{i}{\hbar} \varphi} (i \hbar \partial_{\hat{x}^m} a - \varphi_{\hat{x}^m} a) |dx|^{\frac{1}{2}} \\ \widehat{x}^m : e^{\frac{i}{\hbar} \varphi} a |dx|^{\frac{1}{2}} &\mapsto e^{\frac{i}{\hbar} \varphi} \widehat{x}^m a |dx|^{\frac{1}{2}} \end{aligned}$$

The transition functions corresponding to a change (6.1) act by

$$\begin{aligned} a(x_\beta, \theta) &\mapsto \\ a(g_{\beta\gamma}(x_\gamma), \rho_{\beta\gamma}(x_\gamma, \theta); g_{\beta\gamma}(x_\gamma + \hat{x}) - x_\beta, \rho_{\beta\gamma}(x_\gamma + \hat{x}, \theta + \hat{\theta}) - \rho_{\beta\gamma}(x_\gamma, \theta)) &\times \\ \times |\det \frac{\partial \rho_{\beta\gamma}(x_\gamma + \hat{x}, \theta + \hat{\theta})}{\partial \theta}|^{-1} |\det g'_{\beta\gamma}(x_\gamma + \hat{x})|^{\frac{1}{2}} & \end{aligned} \quad (7.3)$$

One defines also the flat connection

$$\nabla = (\frac{\partial}{\partial x} - \frac{\partial}{\partial \hat{x}}) dx = \sum (\frac{\partial}{\partial x^m} - \frac{\partial}{\partial \hat{x}^m}) dx^m$$

One gets a bundle of π^* jets $\mathcal{R}D_X^{\frac{1}{2}}$ -modules with a flat connection which is compatible with the canonical connection on the jet bundle. We would like to modify this construction as follows. Recall that we have constructed in the proof of proposition 4.9.1 a multiplication on the jet algebra $J_{T^*X}[[\hbar]]$ and a Fedosov connection whose lifting has the curvature $\frac{1}{i\hbar} \omega$. We denote the resulting bundle of algebras by \mathcal{W}_{T^*X} . We would like to get a bundle of \mathcal{W}_{T^*X} -modules with a flat connection which is compatible with the Fedosov connection. To achieve that, we have to modify our module J .

Definition 7.1.1. For a point \mathbf{x} of $L \cap U_\beta$ represented by (x, θ) , put

$$(J_L^H)_{\mathbf{x}} = \{e^{\frac{i}{\hbar} \varphi_\beta(x, \theta; \hat{x}, \hat{\theta})} a(x, \theta; \hat{x}, \hat{\theta}; \hbar)\} / \sim \quad (7.4)$$

where

$$\varphi_\beta(x, \theta; \hat{x}, \hat{\theta}) = \varphi_\beta(x + \hat{x}, \theta + \hat{\theta}) - \varphi_\beta(x, \theta) - \hat{x} \partial_x \varphi_\beta(x, \theta) - \hat{\theta} \partial_\theta \varphi_\beta(x, \theta) \quad (7.5)$$

and

$$i \hbar \partial_{\hat{\theta}} a - \partial_{\hat{\theta}} \varphi_\beta(x, \theta; \hat{x}, \hat{\theta}) \sim 0 \quad (7.6)$$

Define the action of \mathcal{W}_{U_β} on the above space by:

$$\begin{aligned}\widehat{\xi} : e^{\frac{i}{\hbar}\varphi_\beta} a &\mapsto e^{\frac{i}{\hbar}\varphi_\beta} (i\hbar\partial_{\widehat{x}}a - \partial_{\widehat{x}}\varphi_\beta a) \\ \widehat{x} : e^{\frac{i}{\hbar}\varphi_\beta} a &\mapsto e^{\frac{i}{\hbar}\varphi_\beta} \widehat{x}a\end{aligned}\tag{7.7}$$

Lemma 7.1.2. *The transition functions (7.3) define a structure of a bundle of $\mathcal{W}_L \otimes \mathbb{K}$ -modules on J_L^H . The formula*

$$\nabla = \left(\frac{\partial}{\partial x} - \frac{\widehat{\xi}}{i\hbar}\right)dx + \left(\frac{\partial}{\partial \xi} + \frac{\widehat{x}}{i\hbar}\right)d\xi$$

defines a flat connection on J_L^H .

Proof. Let us check how the above definition differs from the one given by (7.2), (7.3). There are two differences, namely the constant term and the linear term of $\varphi(x + \widehat{x}, \theta + \widehat{\theta})$. But these two differences exactly mirror the differences between the two bundles of algebras with connection, namely $(\pi^*\text{jets}\mathcal{R}D_X^{\frac{1}{2}}, \pi^*\nabla_{\text{can}})$ and $(\mathcal{W}_{T^*X}, \nabla)$; they disappear after we modify the bundle and the connection as in 4.9. Indeed,

$$\begin{aligned}\varphi(x + \widehat{x}, \theta + \widehat{\theta}) &= \varphi(x, \theta) + \widehat{x}\varphi_x(x, \theta) + \widehat{\theta}\varphi_\theta(x, \theta) + \varphi(x, \theta; \widehat{x}, \widehat{\theta}) = \\ &= \varphi(x, \theta) + \xi\widehat{x} + \varphi(x, \theta; \widehat{x}, \widehat{\theta})\end{aligned}$$

if the point (x, θ) corresponds to a point of L . After identifying $\pi^*\text{jets}\mathcal{R}D_X^{\frac{1}{2}}$ with W^{fin} (cf. (4.20)), we get a bundle of W^{fin} -modules with a compatible connection; after the gauge transformation $\exp(\frac{1}{i\hbar}\xi\widehat{x})$ is applied to the module, it becomes a bundle of $\mathcal{W} \otimes \mathbb{K}$ -modules. In coordinates, the connection is equal to

$$\nabla = \frac{i}{\hbar}d\varphi + \left(\partial_x - \frac{1}{i\hbar}\widehat{\xi}\right)dx + \left(\partial_\xi + \frac{1}{i\hbar}\widehat{x}\right)d\xi.$$

But $d\varphi = \alpha$ on L ; so, after adding $\frac{1}{i\hbar}\xi dx$ to the connection on W , the first term vanishes.

Example 7.1.3. Let L be given by the equation $\xi^m = \varphi_{x^m}(x)$ on $T^*\mathbb{R}^n$. Then sections of the modified jet bundle are formal expressions

$$e^{\frac{i}{\hbar}(\varphi(x+\widehat{x})-\varphi(x)-\widehat{x}\varphi_x(x))}a(x, \widehat{x})|dx|^{\frac{1}{2}}$$

on which \widehat{x}^m acts by multiplication and ξ^m by $i\hbar\partial_{\widehat{x}^m}$. The connection

$$\left(\partial_x - \frac{1}{i\hbar}\widehat{\xi}\right)dx + \left(\partial_\xi + \frac{1}{i\hbar}\widehat{x}\right)d\xi$$

acts at the level of $a(x, \widehat{x})$ by $(\partial_x - \partial_{\widehat{x}})dx$.

Example 7.1.4. Let L be given by the equation $x^m = -\psi_{\xi^m}(\xi)$ on $T^*\mathbb{R}^n$. Then sections of the modified jet bundle are formal expressions

$$e^{\frac{i}{\hbar}(\widehat{x}\widehat{\theta} + \psi(\xi + \widehat{\theta}) - \psi(\xi) - \widehat{\theta}\psi_{\xi}(\xi))} a(\xi, \widehat{x}, \widehat{\theta}) |d\xi|^{\frac{1}{2}} \bmod \sim$$

where $i\hbar\partial_{\widehat{\theta}}(e^{\frac{i}{\hbar}\dots} a \dots) \sim 0$. The space of such local sections is isomorphic to the space of expressions

$$e^{\frac{i}{\hbar}(\widehat{x}\widehat{\theta} + \psi(\xi + \widehat{\theta}) - \psi(\xi) - \widehat{\theta}\psi_{\xi}(\xi))} a(\xi, \widehat{\theta}) |d\xi|^{\frac{1}{2}}$$

on which W acts, at the level of the factor a , by

$$\begin{aligned} \widehat{\xi}^m : a &\mapsto -i\hbar\partial_{\widehat{\theta}}a \\ \widehat{x}^m : a &\mapsto i\hbar\frac{\partial a}{\partial \widehat{\theta}} + \psi(\xi + \widehat{\theta}) - \psi(\xi) \end{aligned}$$

The connection

$$(\partial_x - \frac{1}{i\hbar}\widehat{\xi})dx + (\partial_{\xi} + \frac{1}{i\hbar}\widehat{x})d\xi$$

acts at the level of $a(\xi, \widehat{\theta})$ by $(\partial_{\xi} - \partial_{\widehat{\theta}})d\xi$.

The above two examples generalize to the case of any special phase function (6.11). From this one deduces

Proposition 7.1.5. *One has for the Lagrangian module V_L :*

$$V_L \xrightarrow{\sim} (J_L^H)^{\nabla} \otimes \mathcal{E}_{\frac{i}{\hbar}\alpha_L}$$

where the first factor in the right hand side stands for the sheaf of horizontal sections of J_L^H and α_L is an \mathbb{R} -valued one-cocycle representing the cohomology class of $\alpha = \xi dx$ on L .

Furthermore, from the explicit formulas for the transition functions one observes the following

Proposition 7.1.6. *The fiber of J_L^H is isomorphic to $\mathbb{C}[[\hbar, \widehat{x}]] \otimes_{\mathbb{C}[[\hbar]]} \mathbb{K}$. If we put $|\widehat{x}| = 1$ and $|\hbar| = 2$ and consider the filtration $F^m = \prod_{p \geq m} \{a \mid |a| = p\}$, then this filtration induces a filtration on J_L^H which is compatible with the similar filtration on W .*

8. THE MAIN STATEMENT

8.1. We have defined a deformation quantization \mathbb{A}_{T^*X} (subsection 4.9) and a sheaf of $\mathbb{A}_{T^*X} \otimes \mathbb{K}$ -modules V_L (section 6). Now, using Darboux-Weinstein theorem, we can identify a neighborhood of L in T^*X with a neighborhood $\overset{0}{T^*}L$ of L in T^*L . We can construct the algebra $\mathbb{A}_{T^*L}^0$ and the module V_L^0 using this identification, and choosing the zero section L as a Lagrangian submanifold of T^*L .

Proposition 8.1.1. *There exists an isomorphism of algebras on T^*L*

$$\mathbb{A}_{T^*X} \xrightarrow{\sim} \mathbb{A}_{T^*L}^0$$

which is canonical up to a canonical inner automorphism.

Indeed, because of the remark in the end of the proof of 4.9.1, one can apply Theorem 4.7.1.

Theorem 8.1.2. *Let us identify the algebras \mathbb{A}_{T^*X} and $\mathbb{A}_{T^*L}^0$ using Proposition 8.1.1. There exists an isomorphism of modules*

$$V_L \xrightarrow{\sim} V_L^0 \otimes_{\mathbb{C}[[\hbar]]} \mathcal{E}_{\frac{i}{\hbar}\alpha_L + \frac{\pi i}{2}\mu_L}$$

where α_L is an \mathbb{R} -valued one-cocycle representing the cohomology class of $\alpha = \xi dx$ on L , μ_L is a \mathbb{Z} -valued one-cocycle representing the Maslov class of L , and $\mathcal{E}_{\frac{i}{\hbar}\alpha_L + \frac{\pi i}{2}\mu_L}$ is the \mathbb{K} -valued local system on L with the transition functions $\exp(\frac{i}{\hbar}\alpha_L + \frac{\pi i}{2}\mu_L)$.

Proof. We have proven that \mathbb{A}_{T^*X} is isomorphic to the algebra of horizontal sections of the bundle of algebras \mathcal{W}_{T^*X} , and there is a compatible isomorphism of V_L to the module of horizontal sections of the bundle of modules J_L^H . Similarly, $\mathbb{A}_{T^*L}^0$ is isomorphic to the algebra of horizontal sections of the bundle of algebras $\mathcal{W}_{T^*L}^0$, and that there is a compatible isomorphism of V_L^0 to the module of horizontal sections of the bundle of modules $J_L^{H,0}$. (The algebra isomorphisms are canonical up to a canonical inner automorphism). Therefore the statement of the theorem follows from

Proposition 8.1.3. 1). *There is a connection-preserving isomorphism of bundles of algebras on T^*L*

$$\mathcal{W}_{T^*X} \xrightarrow{\sim} \mathcal{W}_{T^*L}^0$$

which is canonical up to a conjugation by a canonical invertible horizontal element.

2). *There is a connection-preserving isomorphism of bundles of modules*

$$J_L^H \xrightarrow{\sim} J_L^{H,0} \otimes_{\mathbb{C}[[\hbar]]} \mathcal{E}_{\frac{\pi i}{2}\mu_L}$$

compatible with the above isomorphism of bundles of algebras.

Proof of Proposition. All our local phase functions will be of the special form (6.11). Start with the transition functions for the bundle of algebras \mathcal{W}_{T^*X} ;

$$\tilde{G}_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow K \rightarrow \tilde{G}_{\geq 0}$$

Similarly, consider the lifted transition functions $\tilde{G}_{\alpha\beta}^0$ for the bundle $\mathcal{W}_{T^*L}^0$. The group K and its embedding to $\tilde{G}_{\geq 0}$ are explained in the end of 4.9. The lifted Fedosov connection, in local coordinates, is given by the formula

$$(\partial/\partial x - \widehat{\xi}/i\hbar)dx + (\partial/\partial \xi + \widehat{x}/i\hbar)d\xi \quad (8.1)$$

Here and below we call the connection given by this formula *the canonical connection*.

Now replace $\tilde{G}_{\alpha\beta}$ by an equivalent set of transition functions

$$\tilde{G}_{\alpha\beta}^{\text{new}} = \tilde{H}_\alpha \tilde{G}_{\alpha\beta} \tilde{H}_\beta^{-1}$$

where

$$\tilde{H}_\alpha : U_\alpha \rightarrow \tilde{G}_{\geq 0}$$

are defined as follows.

Let $L \cap U_\alpha$ be given by (6.11). Define

$$\sigma_\alpha = \exp \frac{i}{\hbar} (F(x_1 + \widehat{x}_1, \xi_2 + \widehat{\xi}_2) - F(x_1, \xi_2) - F_{x_1}(x_1, \xi_2)\widehat{x}_1 - F_{\xi_2}(x_1, \xi_2)\widehat{\xi}_2)$$

in $\tilde{G}_{\geq 0}$. Now consider the local coordinate change

$$(x_1, x_2) \mapsto (x_1, \xi_2); \quad (\xi_1, \xi_2) \mapsto (\xi_1, -x_2)$$

and a symplectic transformation (the partial Fourier transform)

$$\mathbf{F}_\alpha : \widehat{x} \mapsto (\widehat{x}_1, \widehat{\xi}_2); \quad \widehat{\xi} \mapsto (\widehat{\xi}_1, -\widehat{x}_2)$$

We fix liftings $\tilde{\mathbf{F}}_\alpha$ to $\widetilde{\text{Sp}}$ (counterclockwise rotation in $(\widehat{x}_2, \widehat{\xi}_2)$ space). Put

$$\tilde{H}_\alpha = \tilde{\mathbf{F}}_\alpha \sigma_\alpha$$

Note that the above formula is precisely the Maslov canonical operator, defined here at the jet level.

We get the new bundle of algebras, which we denote by $\mathcal{W}_{T^*X}^{\text{new}}$. The connection on J_L^H is given by the same formula as the canonical connection. The action of $\mathcal{W}_{T^*X}^{\text{new}}$ on J_L^H is, in our new local coordinates, the standard one: \widehat{x} acts by multiplication, and $\widehat{\xi}$ by $i\hbar \frac{\partial}{\partial \widehat{x}}$. Note that, because of this, the transition functions of the module J_H^L determine the transition functions of the bundle of algebras $\mathcal{W}_{T^*X}^{\text{new}}$. The same is true about $J_L^{H,0}$ and $\mathcal{W}_{T^*L}^0$.

We claim that:

1) The transition functions $\tilde{G}_{\beta\gamma}^{\text{new}}$ take values in the subgroup P (cf. Lemma 4.3.2; note also that K is a subgroup of P).

2) The image of $\tilde{G}_{\beta\gamma}^{\text{new}}$ in P/N is equal to the image of $\mu_{\beta\gamma} \tilde{G}_{\beta\gamma}^0$.

Here $\mu_{\beta\gamma}$ is the cocycle representing the Maslov class, as in (3.9), with values in $\mathbb{Z} \subset \tilde{G}_{\geq 0}$.

3) The transition functions of the bundle of modules J_L^H are equal to $\exp(\alpha_{\beta\gamma})w(\tilde{G}_{\beta\gamma}^0)$ where $\alpha_{\beta\gamma}$ is the specific cocycle representing the 1-cohomology class α of L as in (6.25)

$$w : P \rightarrow \text{Aut}(\mathbb{C}[[\hat{x}, \hbar]])$$

is the restriction of the degenerate Weil representation

$$w : \tilde{G}_{\geq 0} \rightarrow \text{Aut}(\hat{V}_{\text{Weil}}^0)$$

to the subgroup preserving the subspace $V_{T=0} = \mathbb{C}[[\hat{x}, \hbar]]$.

To prove the claim, observe first the following.

A) The transition functions $\tilde{G}_{\beta\gamma}^{\text{new}}$ and $\tilde{G}_{\beta\gamma}^0$ are the same if they correspond to a coordinate change $g_{\alpha(\beta)\alpha(\gamma)}$ on X , and U_β, U_γ are such that the projections of $U_\beta \cap L$ and $U_\gamma \cap L$ to the base are bijective. Similarly, the transition functions of J_L^H and $J_L^{H,0}$ are the same, and are the image of the above under w .

B) The same is true modulo N if the transition functions $G_{\beta\gamma}^{\text{new}}$ correspond to a change of subdivision $x = (x_1, x_2)$. This follows from the formula (6.23), or rather from its version for the power series (cf. [K]). More precisely, the transition functions $\tilde{G}_{\beta\gamma}^{\text{new}}$ are given by

$$\mu_{\beta\gamma} \exp\left(\frac{1}{i\hbar} \sum \hbar^{h_1(\Gamma)} c_\Gamma\right)$$

where Γ are all connected graphs; the sum of the terms with $h_1(\Gamma) = 0, 1$ is exactly the transition functions $\tilde{G}_{\beta\gamma}^0$. Similarly, the transition functions of the module J_L^H are given by

$$\exp\left(\frac{\pi i}{2} \mu_{\beta\gamma}\right) \left(\frac{1}{i\hbar} \sum \hbar^{h_1(\Gamma)} c_\Gamma\right)$$

and the transition functions of the module $J_L^{H,0}$ are given by the sum of the terms with $h_1(\Gamma) = 0, 1$. Again, we see directly that the transition functions of the bundle of modules are the image under w of the transition function of the bundle of algebras.

C) It remains to compare our transition functions for the rest of coordinate changes, namely, when the coordinate change corresponds to a coordinate change on the base and the subdivision $x = (x_1, x_2)$ has $n_2 > 0$. Observe that all the transition functions that we are considering are given by universal formulas in terms of two jets of coordinate systems on the base, a jet of a Lagrangian, and two subdivisions $x = (x_1, x_2)$. On an open dense subset, the transition functions of the type C) can be expressed through the transition functions of types A),

B). But, because of the above arguments, all the equalities that we are proving are true on an open dense subset; therefore, they are true everywhere.

We see now that the transition functions $\tilde{G}_{\beta\gamma}^{\text{new}}$ and $\tilde{G}_{\beta\gamma}^0$ differ by a Čech one-cocycle with values in N , invariant under the canonical connection $(\partial/\partial x - \hat{\xi}/i\hbar)dx + (\partial/\partial \xi + \hat{x}/i\hbar)d\xi$. But it is easy to see that any such cocycle is cohomologous to the identity. In fact, finding a zero-cochain of which it is a coboundary reduces to an iterative procedure whose individual steps are to trivialize a Čech one-cocycle with coefficients in a sheaf of smooth sections of a C^∞ vector bundle.

8.2. The main statement in the Fedosov form. We finish by identifying the Maslov-Hörmander construction in deformation quantization in Fedosov terms. Our first goal is to determine the structure of the associated graded module $\text{gr}_F J_L^H$.

8.2.1. The flat bundle $(W/WT_L^\perp) \otimes |\Omega_L|^{\frac{1}{2}}$. For any symplectic M and any Lagrangian submanifold L , let T_L^\perp be the conormal bundle of L , viewed as a subbundle of the Weyl bundle W_L . Obviously, W/WT_L^\perp is a W -module. Let ∇ be a Fedosov connection compatible with L . Choose a flat connection on the bundle of half-densities $|\Omega_L|^{\frac{1}{2}}$. The tensor product $(W/WT_L^\perp) \otimes |\Omega_L|^{\frac{1}{2}}$ becomes a W -module with a flat connection which is compatible with the Fedosov connection on W .

The following is easy to see from the explicit definition of J_L^H .

Proposition 8.2.1. *There is an isomorphism of bundles of $W \otimes \mathbb{K}$ -modules*

$$\text{gr}_F(J_L^H \otimes_{\mathbb{C}[[\hbar]]} \mathcal{E}_{\frac{i}{\hbar}\alpha_L + \frac{\pi i}{2}\mu_L}^{-1}) \xrightarrow{\sim} (W/WT_L^\perp \otimes |\Omega_L|^{\frac{1}{2}})$$

where α_L is an \mathbb{R} -valued one-cocycle representing the cohomology class of $\alpha = \xi dx$ on L , μ_L is a \mathbb{Z} -valued one-cocycle representing the Maslov class of L , and $\mathcal{E}_{\frac{i}{\hbar}\alpha_L + \frac{\pi i}{2}\mu_L}$ is the \mathbb{K} -valued local system on L with the transition functions $\exp(\frac{i}{\hbar}\alpha_L + \frac{\pi i}{2}\mu_L)$.

Theorem 8.2.2. 1) *The deformed algebra \mathbb{A}_{T^*X} is isomorphic to the algebra of horizontal sections of the bundle W . This isomorphism is canonical up to a canonical inner automorphism.*

2) *Under the identification from the statement 1), the Lagrangian module V_L^H is isomorphic to the sheaf of horizontal sections of $(W/WT_L^\perp \otimes |\Omega_L|^{\frac{1}{2}}) \otimes_{\mathbb{C}[[\hbar]]} \mathcal{E}_{\frac{i}{\hbar}\alpha_L + \frac{\pi i}{2}\mu_L}$.*

Statement 1) follows from Theorem 4.7.1. Statement 2) follows from Proposition 8.1.3 and the following

Proposition 8.2.3. *The bundles of W_L -modules with connections are isomorphic:*

$$J_L^H \xrightarrow{\sim} (W/WT_L^\perp \otimes |\Omega_L|^{\frac{1}{2}}) \otimes_{\mathbb{C}[[\hbar]]} \mathcal{E}_{\frac{i}{\hbar}\alpha_L + \frac{\pi i}{2}\mu_L}$$

Proof. Proposition 8.1.3 reduces this statement to the case when L is the zero section of the cotangent bundle, where it is easy to see explicitly.

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MATHEMATICAL INSTITUTE, COPENHAGEN UNIVERSITY, UNIVERSITATSPARKEN,
2100 COPENHAGEN, DENMARK

E-mail address: rnest@math.ku.dk

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON,
IL, 60208, USA

E-mail address: tsygan@math.nwu.edu